Compact extra dimensions in cosmologies with $f(T)$ structure

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The presence of compact extra dimensions in cosmological scenarios in the context of $f(T)$-like gravities is discussed. For the case of toroidal compactifications, the analysis is performed in an arbitrary number of extra dimensions. Spherical topologies for the extra dimensions are then carefully studied in six and seven spacetime dimensions, where the proper vielbein fields responsible for the parallelization process are found.

arXiv:1304.1912v1 [gr-qc] 6 Apr 2013

I. INTRODUCTION

In 1928 Einstein proposed an equivalent formulation of General Relativity (GR) nowadays known as Teleparallel equivalent of General Relativity (TEGR) [1, 2]. In this theory, the Weitzenböck connection is used instead of the Levi-Civita connection to define the covariant derivative. Weitzenböck connection have no null torsion, however is curvatureless which implies that this formulation of gravity exhibits only torsion. In D spacetime dimensions the dynamical fields are the D linearly independent vielbeins and the torsion tensor is formed solely by them and first derivatives of these objects. The Lagrangian density, which will be noted T hereafter, is constructed from this torsion tensor assuming the invariance under general coordinate transformations and local Lorentz transformations, along with demanding the Lagrangian density to be quadratic in the torsion tensor [2]. In recent years an extension of the above Lagrangian was constructed in [3–6], making the Lagrangian density a function of the scalar torsion T, the so-called $f(T)$ gravity. Much attention has been focused on this extended $f(T)$ gravity theory recently, because it exhibits interesting cosmological implications, as witnessed in the study of missing matter problems and also as a new mechanism to explain the late acceleration of the universe based on a modification of the gravitational theory, instead of introducing an exotic context of matter (Dark energy problem) [7–18]. Also, it is believe that $f(T)$ gravity could be a reliable approach to address the shortcomings of general relativity at high energy scales [19]. For instance, in [4] a Born-Infeld $f(T)$ gravity Lagrangian was used to cure the physically inadmissible divergencies occurring in the Big Bang of the standard cosmology, rendering the spacetime geodesically complete and powering an inflationary stage without the introduction of an inflaton field.

We are interesting now in studying this alternative theory of gravity in a higher dimensional context, and to extract the basic features of the cosmological behavior due to this extension. The conception of extra dimensions in physical theories have a long history that begins with the original ideas of Kaluza and Klein [20] and finds a new realization nowadays within the modern string theory [21, 22]. One of the main motivations to study these higher dimensional models is the chance of unification of the fundamental interactions of nature. In most of extra dimensional scenarios the additional dimensions are assumed compactified on a very small (unobservable) internal submanifold while the other spacetime dimensions constitute the 4-dimensional observable universe. Another mechanism of dimensional reduction studied in cosmological GR scenarios correspond to the dynamical compactification [23–25]. In this case, the internal dimensions evolve in time to very small scales while the external dimensions expand, and so, the observable universe becomes effectively four dimensional. This type of reduction was first studied by Chodos and Detweiler in Ref. [23], where they have considered vacuum Einstein equations in five spacetime dimensions. This results were extended latter, and a full classification of homogeneous eleven dimensional cosmologies can be found in [26]. Dynamical reduction in multidimensional Bianchi type I models was studied in [27] and a study of multidimensional cosmological models as dynamical systems with topology $\mathrm{FRW}^4 \times T^{D-4}$ (here T^{D-4} refers to the $(D-4)$-torus) was performed in [28, 29].

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Quite more recently, and in a quite different context, GR in the presence of a large number of extra dimensions was carefully studied in [30, 31].

The specific topic of extra dimensions have been barely worked out in $f(T)$ theories, and it remits to some black hole solutions of $f(T)$ gravity in arbitrary spacetime dimensions [32]. In this work we investigate for the first time cosmological solutions in the context of $f(T)$ gravity in higher dimensions, considering that extra dimensions are compactified in different ways. We consider both toroidal and spherical topologies for the internal dimensions. With this purpose in mind we proceed first to obtain the proper parallel vector fields for the different topologies under consideration. This topic is hard by itself, and constitutes the starting point of any model describing extra dimensions in gravitational theories with absolute parallelism, $f(T)$ gravity being one of them. In particular we consider five, six and seven dimensional models and study all the possible compactifications of the extra dimensions using the right vielbein field that parallelizes the spacetime in each of these cases, and we proceed later to obtain the corresponding field equations. Then we show in general lines, that in some limit cases, simple analytical solutions describe physically relevant spacetimes representing different stages of the cosmic evolution.

The paper is organized as follows. In section II, we present a brief review of the fundamentals of $f(T)$ theories. In section III we study toroidal compactification for an arbitrary number of extra dimensions and in section IV we study spherical compactifications with emphasis in six and seven spacetime dimensions. Finally, in section V we display the conclusions.

II. FUNDAMENTALS OF $f(T)$ THEORIES

The extended gravitational schemes with absolute parallelism ($f(T)$ theories), take as starting point the teleparallel equivalent of General Relativity (TEGR). We will just summarize here the basic elements needed in order to elaborate the ideas of the present work, leaving the details for the reader. For a thorough introduction to $f(T)$ gravity as well as to their mathematical basis the reader can consult, for instance, Refs. [33–37].

The spirit of the equivalence between the Riemann and Weitzenböck formulations of GR can be summarized in the equation

$$T \;=\; -R + 2\, e^{-1}\, \partial_\nu (e\, T_\sigma{}^{\sigma\nu}\,)\ . \tag{1}$$

In the left hand side of Eq. (1) we have the so called Weitzenböck invariant

$$T \;=\; S^\rho{}_{\mu\nu}\; T_\rho{}^{\mu\nu}\,, \tag{2}$$

where $T_\rho{}^{\mu\nu}$ are the components (in a coordinate basis) of the torsion two form $T^a = de^a$ coming from the Weitzenböck connection $\Gamma^\lambda_{\nu\mu} = e^\lambda_a\, \partial_\nu e^a_\mu$, and $S_{\lambda\mu\rho}$ is defined according to

$$S^\rho{}_{\mu\nu} = \frac{1}{4}\,(T^\rho{}_{\mu\nu} - T_{\mu\nu}{}^\rho + T_{\nu\mu}{}^\rho) + \frac{1}{2}\;\delta^\rho_\mu\; T_{\sigma\nu}{}^\sigma - \frac{1}{2}\;\delta^\rho_\nu\; T_{\sigma\mu}{}^\sigma. \tag{3}$$

Actually, T is the result of a very specific quadratic combination of irreducible representations of the torsion tensor under the Lorentz group $SO(1,3)$ [38]. Equation (1) simply say that the Weitzenböck invariant T differs from the scalar curvature R in a total derivative, hence both conceptual frameworks result totally equivalent at the time of describing the dynamics of the gravitational field.

$f(T)$ gravity can be viewed as a natural extension of Einstein gravity, and is ruled by the action in D spacetime dimensions

$$S = \frac{1}{2\kappa}\int d^D x\, e\, [f(T) + L_{matter}]\ \ , \tag{4}$$

being $e = \sqrt{det(g_{\mu\nu})}$. Of course GR is contained in (4) as the particular case when $f(T) = T$. The dynamical equations in $f(T)$ theories, for matter coupled to the metric in the usual way, are

$$\Big(e^{-1}\,\partial_\mu (e\; S_a{}^{\mu\nu})\; +\; e^\lambda_a\; T^\rho{}_{\mu\lambda}\; S_\rho{}^{\mu\nu}\Big)\; f'(T)\; +S_a{}^{\mu\nu}\; \partial_\mu(T)\; f''(T) - \frac{1}{4} e^\nu_a\; f(T) \;=\; -4\pi G\; e^\lambda_a\; T_\lambda{}^\nu\ , \tag{5}$$

where the prime means derivative with respect to T and $T_\lambda{}^\nu$ is the energy-momentum tensor. It is worth insisting in the fact that equations (5) are second order differential equations for the vielbein components. This implies a genuine advantage compared with other deformed gravitational schemes as, for instance, the popular $f(R)$ gravities. This

goodness, however, brings consequences: the lack of local Lorentz invariance. If we perform to a given solution (let's say e^b), a local Lorentz transformation $e^a \longrightarrow e^{a'} = \Lambda_b^{a'}(x)\ e^b$, we will find that $e^{a'}$ is not a solution of the field equations (5) anymore. The reason for this lack of local invariance is simple: under a local transformation the scalar (2) changes according to $T \to T' = T+$ surface term. So this surface term, which is harmless when $f(T) = T$, remains inside the function f, ruining thus the local invariance of the theory. This is because the theory picks up preferred referential frames which constitute the autoparallel curves of the given manifold. In other words, the field equations (5) determine the full components of the vielbein and not just those of the metric tensor, related with the vielbein by means of

$$g_{\mu\nu}(x) = \eta_{ab}\ e_\mu^a(x)\ e_\nu^b(x). \tag{6}$$

An important fact sometimes omitted in the literature concerns the coupling with the matter fields. If we assume that the matter action depends only on the metric (and not on the whole vielbein), then the non fermionic matter fields are unable to sense the lack of local Lorentz invariance by virtue of the invariant character of expression (6) under such transformations. Then, as far as coupling to (non spinning) matter concerns, the lack of local invariance should not be problematic. The effects of the additional degrees of freedom that certainly exist in $f(T)$ theories as a consequence of the breaking of Lorentz invariance (see the analysis performed in [39]), must be found beyond the unspinning matter. Actually, it was found that on the flat FRW background with a scalar field, linear perturbation up to second order does not reveal any extra degree of freedom at all [35]. It is fair to say though that the nature of the additional degrees of freedom remains unknown.

III. TOROIDAL COMPACTIFICATIONS IN $D-4$ INTERNAL DIMENSIONS

In this section we will study a D-dimensional cosmological model inside $f(T)$-gravity theory. We will consider the simplest case where the adittional internal dimensions are compactified in a torus, and in the next section we will discuss others topologies for the extra dimensions. The external spacetime is assumed to be spatially flat in accordance with the current experimental evidence. Thus, the topology of our cosmological model is described by $R \times R^3 \times S^1 \times ... \times S^1$, where $S^1 \times ... \times S^1$ correspond to the $D-4$ dimensional torus. Under these considerations the metric is written as

$$ds^2 = dt^2 - a_0^2(t)(dx^2 + dy^2 + dz^2) - \sum_{n=1}^{D-4} a_n(t)\, dX_n^2, \tag{7}$$

where $a_0(t)$ denotes the scale factor of the external space and $a_n(t)$ $(n = 1, ..., D-4)$ the scale factors for the internal dimensions X_n. In accordance with the present observations we know that the internal compact dimensions must be very small in order to be undetectable, so we are mainly interested in solutions where all the radius of the internal dimensions $a_n(t)$ $(n = 1, ..., D-4)$ become small as the cosmic time evolves. To proceed, we need first to find the correct parallelization of the spacetime under consideration. For the metric (7), and due to the fact that the circle S^1 is trivially parallelizable, a vielbein that parallelizes the spacetime is the simple diagonal one

$$e_\mu^a = diag\,(1, a_0(t), a_0(t), a_0(t), a_1(t), ..., a_{D-4}(t))\,. \tag{8}$$

Actually, we can think about the vielbein (8) as follows: unwrap the periodic coordinates X_n, and take the submanifold $t = constant$. After rescaling the coordinates according to

$$\begin{aligned}
&x \to x' = a_0\, x \\
&y \to y' = a_0\, y \\
&z \to z' = a_0\, z \\
&X_n \to X_n' = a_n\, X_n,
\end{aligned}$$

we just get $D-1$ Euclidian space. For this space, it is clear that the autoparallel lines are just the straight lines which can be generated by the basis ∂_i $(i : 1, ...D-1)$, whose dual cobasis is dx_i. So, up to a time dependent conformal factor, the frames describing the autoparallel lines are dx_i, and then the whole spacetime is parallelized as indicated in (8).

In order to obtain the motion equations, we shall work in the co-moving system, where the energy-momentum tensor for a perfect fluid reads

$$T^{\mu}_{\nu} = diag(\rho, -p_0, -p_0, -p_0, -p_1, ..., -p_{D-4}). \tag{9}$$

In this way, the energy conservation equation takes the form

$$\dot{\rho} + (3H_0 + \sum_{n=1}^{D-4} H_n)\rho + 3H_0 p_0 + \sum_{n=1}^{D-4} H_n p_n = 0, \tag{10}$$

where a dot means derivative with respect to time and $H_0 = \dot{a}_0/a_0$ and $H_n = \dot{a}_n/a_n$, are the Hubble parameters of the submanifolds with the scale factors $a_0(t)$ and $a_n(t)$, respectively. We can compute now the Weitzenböck invariant (2), which turns out to be

$$T = -6\left(H_0^2 + H_0 \sum_{n=1}^{D-4} H_n + \frac{1}{3}\sum_{n=1}^{D-4} H_n\, H_{n+1}\right). \tag{11}$$

The initial value equation, i.e., the equation coming from the variation of the action respect the e^0_0 component of the vielbein results

$$f - 2\,T f' = 16\pi G\rho. \tag{12}$$

On the other hand, the action variation respect the spatial sector of the vielbein, leads to the following equations

$$2f'\left(3H_0^2 + 2\dot{H}_0 + \sum_{n=1}^{D-4}(\dot{H}_n + 2H_0H_n + H_n^2) - \frac{T}{2}\right) + 2f''\dot{T}\left(2H_0 + \sum_{n=1}^{D-4} H_n\right) + f(T) = -16\pi G p_0, \tag{13}$$

$$2f'\left(6H_0^2 + 3\dot{H}_0 + \sum_{n=1, n\neq b}^{D-4}(\dot{H}_n + H_0H_n + H_n^2) - \frac{T}{2}\right) + 2f''\dot{T}\left(3H_0 + \sum_{n=1, n\neq b}^{D-4} H_n\right) + f(T) = -16\pi G p_b. \tag{14}$$

Note that this last expression actually contains $D-4$ equations. The full system of equations to be solved is given by Eqs. (12)-(14) with T given by (11). They contain the conservation equation (10) self consistently.

Now we will investigate some simple limit cases, in order to analyze the behavior of the scale factors. In all of this section, we will consider that the Hubble parameters of the extra dimensions are equal ($H_1 = ... = H_{D-4} \equiv H_1$) and the content of matter in the spacetime is assumed to be dust matter, i.e. $p_0 = p_n = 0$. In this way, Eqs. (11), (13), and (14) are simplified to

$$T = -6H_0^2 - 6\,(D-4)\,H_0H_1 - (D-4)\,(D-5)\,H_1^2, \tag{15}$$

$$2f'\left(3H_0^2 + 2\dot{H}_0 + (D-4)(\dot{H}_1 + 2H_0H_1 + H_1^2) - \frac{T}{2}\right) + 2f''\dot{T}\,(2H_0 + (D-4)H_1) + f(T) = 0, \tag{16}$$

$$2f'\left(6H_0^2 + 3\dot{H}_0 + (D-5)(\dot{H}_1 + H_0H_1 + H_1^2) - \frac{T}{2}\right) + 2f''\dot{T}\,(3H_0 + (D-5)H_1) + f(T) = 0. \tag{17}$$

First, we will analyze the limit case when both Hubble parameters, H_0 and H_1, are constants, therefore from Eq. (15) we know that T must be a constant too, and the above equations reduce to a more handle form

$$2f'\left(3H_0^2 + (D-4)(2H_0H_1 + H_1^2) - \frac{T}{2}\right) + f(T) = 0, \tag{18}$$

$$2f'\left(6H_0^2 + (D-5)(H_0H_1 + H_1^2) - \frac{T}{2}\right) + f(T) = 0. \tag{19}$$

The system (12), (18) and (19) admit the exact solution

$$H_1 = -\frac{6H_0}{\sqrt{(D-3)^2+12}-(D-3)}. \tag{20}$$

Having imposed the constancy of H_0 and H_1, we will see under what conditions the initial value and energy conservation equations are satisfied. In this case the energy conservation equation can be written as

$$\dot{\rho} + (3H_0 + (D-4)H_1)\rho = 0. \tag{21}$$

Using the fact that $H_0 = \dot{a}_0(t)/a_0(t)$ and $H_1 = \dot{a}_1(t)/a_1(t)$, the energy density is straightforward integrated from the above expression, leading to

$$\rho = \frac{C}{a_0^3(t)a_1^{D-4}(t)}, \tag{22}$$

where C is an integration constant. The initial value equation is satisfied if ρ is constant (due to the constancy of T), which implies that the term $a_0^3\,(t)\,a_1^{D-4}\,(t)$ must be constant. This condition actually means that the total volume of the higher dimensional space remain constant in time, this ensures that a dynamical contraction of the internal dimensions yields an expanding external space. From Eq. (21) we obtain

$$H_0 = -\frac{(D-4)}{3}H_1. \tag{23}$$

This equation together with Eq. (20) are only satisfied for $D=5$ yielding $a_0(t) = a_0(0)e^{H_0t}$ and $a_1(t) = a_1(0)e^{-3H_0t}$, if $\rho \neq 0$. In turn for $\rho = 0$ (with H_0 and H_1 constants), Eq. (21) is identically satisfied and thus we don't have the constraint given by Eq. (23), therefore in this case we have solutions in all dimensions. On the other hand, from Eq. (20) we obtain

$$a_0(t) = a_0(0)e^{H_0t}, \tag{24}$$

$$a_1(t) = a_1(0)Exp\left(-\frac{6H_0}{\sqrt{(D-3)^2+12}-(D-3)}\,t\right). \tag{25}$$

The solutions analyzed above describe a de Sitter expansion of the external space, while the internal space decreases exponentially.

Another limit case worth to mention is when the Hubble parameters of the extra dimensions vanish ($H_1 = ... = H_{D-4} = 0$), which means that the internal space have a constant volume that should be very small. This represents a very simple model describing the asymptotic evolution of the additional dimensions to a final constant volume. It is easy to see that no solution of such type exist in this context. We will see latter in section IV that this kind of behavior for the extra dimensions is possible in some spherical compactifications.

Finally, lets briefly comment about some additional exact solutions occurring in $D=5$. Irrespective of the functional form of $f(T)$, taking $\rho = 0$ and non constant Hubble parameters in Eqs. (11), (13), and (14) we get

$$T = -6\left(H_0^2 + H_0H_1\right), \tag{26}$$

$$2f'\left(3H_0^2 + 2\dot{H}_0 + \dot{H}_1 + 2H_0H_1 + H_1^2 - \frac{T}{2}\right) + 2f''\dot{T}\,(2H_0 + H_1) + f(T) = 0, \tag{27}$$

$$2f'\left(6H_0^2 + 3\dot{H}_0 - \frac{T}{2}\right) + 2f''\dot{T}\,(3H_0) + f(T) = 0. \tag{28}$$

From Eq. (12), we obtain

$$\frac{f}{2f\prime} = T. \tag{29}$$

Therefore, T is a constant that depends on $f(T)$, and replacing it in Eq. (28), we have

$$\dot{H}_0 + 2H_0^2 + \frac{T}{6} = 0, \qquad H_1 = -\frac{T}{6H_0} - H_0, \tag{30}$$

where we have used (26). In this way we can distinguish three cases which depend on the sign of T.

- Case $T < 0$. Eq. (30) tell us in this case that the Hubble parameter H_0 yields

$$H_0 = \frac{1}{2}\sqrt{\frac{-T}{3}}\, tanh\left(\sqrt{\frac{-T}{3}}(t+B)\right), \tag{31}$$

where B is an integration constant. It is worth of mention that this solution only exists in five dimensions, and in this case both scale factors, which are given by

$$a_0\,(t) = \alpha \cosh^{1/2}\left(\sqrt{\frac{|T|}{3}}\,(t+B)\right), \tag{32}$$

$$a_1\,(t) = \beta \sinh\left(\sqrt{\frac{|T|}{3}}\,(t+B)\right)\cosh^{-1/2}\left(\sqrt{\frac{|T|}{3}}\,(t+B)\right), \tag{33}$$

where α and β are constant, increase in time. This case, thus, seems to be unprovided of physical significance.

- Case $T > 0$. Now H_0 is given by

$$H_0 = -\frac{1}{2}\sqrt{\frac{T}{3}}\tan\left(\sqrt{\frac{T}{3}}\,(t+B)\right), \tag{34}$$

and the scale factors by

$$a_0\,(t) = \alpha \cos^{1/2}\left(\sqrt{\frac{T}{3}}\,(t+B)\right), \tag{35}$$

$$a_1\,(t) = \beta \sin\left(\sqrt{\frac{T}{3}}\,(t+B)\right)\cos^{-1/2}\left(\sqrt{\frac{T}{3}}\,(t+B)\right). \tag{36}$$

In order to be physically admissible, at least as a model of the early stages of the cosmic evolution, we have to bound the proper time to the interval $-\frac{\pi}{2}\sqrt{\frac{3}{T}} \leq t \leq 0$, having taken $B = 0$. In this way a_0 results an increasing function of time, while a_1 decreases from infinite at $t = -\frac{\pi}{2}\sqrt{\frac{3}{T}}$ to a null value at $t = 0$.

- Case $T = 0$. Finally, when the torsion scalar vanishes the scale factors are given by

$$a_0^2\,(t) = 2t + B, \tag{37}$$

$$a_1^2\,(t) = \frac{1}{2t+B}. \tag{38}$$

This behavior for the scale factors shows a suppression of the extra dimensions as the bulk expands linearly in time, and then, constitute a physical admissible simple model of the early Universe. As in the former cases with $T < 0$ and $T > 0$, the solutions are independent of the specific form of the function $f(T)$. We actually included them here as simple examples of exact states of the theory, and we hope that they could serve as a seed for more realistic solutions.

IV. SPHERICAL COMPACTIFICATIONS IN $D-4$ INTERNAL DIMENSIONS

In theories relaying on absolute parallelism, the structure of the parallel vector fields describing the additional compact dimensions is highly non trivial. In this section we investigate further this issue and discuss a number of topics which are mandatory in order to understand the nature of the additional compact dimensions when they are other than the simple toroidal compactifications discussed before. In particular, we shall focus now in extra compact dimensions with topology S^n, $n>1$.

As was discussed in the last section, the circle S^1 is trivially paralellizable. A remarkable result due to Stiefel states that every orientable three dimensional manifold is also parallelizable [40]. Additionally, Kervaire and Milnor [41] have shown that, apart from these two, the only other parallelizable sphere is S^7. For our purposes this means that extra dimension spherical compactifications other than these will possess a rather complicated parallel vector field structure. In order to be aware of the subtleties involved let us consider first two extra dimensions spherically compactified. In this case we have the metric

$$ds^2 = dt^2 - a_1^2(t)(d\theta^2 + \sin^2\theta\, d\phi^2) - a_0^2(t)(dx^2+dy^2+dz^2), \tag{39}$$

where (θ, ϕ) are standard spherical coordinates on the 2-sphere. It is intuitively obvious that the 6−dimensional manifold described by (39) results parallelizable. As a matter of fact, the spacetime topology is $\mathcal{M}_3 \times R^3$, where $\mathcal{M}_3 \approx R \times S^2$, so the full spacetime manifold can be written as a product of two 3−dimensional, orientable (i.e., parallelizable) manifolds. However, is quite important to realize that the parallel vector fields of $\mathcal{M}_3$ will be highly non trivial because S^2 itself is not parallelizable. If we arrange the components of the vielbein in a 6×6 matrix e^a_μ, schematically we will have

$$e^a_\mu = \left(\begin{array}{c|ccc} \mathbb{M}_3 & & \mathbb{O} & \\ \hline & a_0 & 0 & 0 \\ \mathbb{O} & 0 & a_0 & 0 \\ & 0 & 0 & a_0 \end{array}\right) \tag{40}$$

where $\mathbb{O}$ is the 3×3 null matrix and $\mathbb{M}_3$ is the 3×3 matrix representing the parallel *triad* field of $\mathcal{M}_3$. The main point we want to emphasize is that $\mathbb{M}_3$ *cannot* be diagonal. If were so, it would mean that the global vector fields which perform the paralellization of $\mathcal{M}_3$ are just $e^1 = dt$, $e^2 = a_1(t)d\theta$ and $e^2 = a_1(t)\sin\theta\, d\phi$. However, this is impossible by the hairy ball theorem (see, for instance [42]). Even more, it is clear that $\mathbb{M}_3$ cannot have the *block* form

$$\mathbb{M}_3 = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{\mathbb{M}_2} \end{array}\right), \tag{41}$$

either, where $\mathbb{M}_2$ is some 2×2 matrix. In this case, we would have that $\mathbb{M}_2$ is related with the matrix $a_1(t)diag(d\theta, \sin\theta\, d\phi)$ by means of a local rotation. However, again, it is impossible to find a global basis for $\mathcal{T}(S^2)$, the tangent bundle of S^2. This means that the matrix $\mathbb{M}_3$ should be related with $diag(dt, a_1(t)d\theta, a_1(t)\sin\theta\, d\phi)$ by a Lorentz boost, and so, that the structure (41) is not correct. Actually, we should mention that the proper (auto-parallel) vector fields (40) corresponding to the metric (39) are unknown at the present, and the task of finding the parallelization of $\mathcal{M}_3$ remains as an open problem. Fortunately the parallelization of the manifold (39) can be found in a somewhat different manner, and this will be matter of the paragraphs below. There we will find that the correct vector fields have the structure

$$e^a_\mu = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & & \\ 0 & \multicolumn{2}{c}{\mathbb{M}_5} \end{array}\right), \tag{42}$$

where $\mathbb{M}_5$ is the matrix representing the parallelization of the five dimensional manifold $S^2 \times R^3$.

As in the six dimensional case, if one consider $D=7$ the problem can be fully understood and this will also be explained in detail below. For extra dimensions higher than three one need to be more patient. If we take four extra dimensions, besides the trivial toroidal compactifications of the last section, we are lead to the topologies $S^1 \times S^3$,

$S^2 \times S^2$ or S^4. Excepting the first case, where the parallelization is provided by a simple extension of the procedure to be explained below for the S^3 compactification, the parallelization will be highly non trivial because S^2 and S^4 do not admit a global basis. This problem turns out to be even more difficult as the dimension increases, for the number of possible compactifications increases as $D-4$, the number of extra dimensions.

A. Complete characterization of $D = 7$

As a working example let us consider now the various possibilities arising from the case $D = 7$. The first of them was worked in the last section, where toroidal compactifications were extensively discussed. Now we will consider the 7−dimensional metric

$$ds^2 = dt^2 - a_1^2(t)[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)] - a_0^2(t)(dx^2 + dy^2 + dz^2), \tag{43}$$

where we have supposed three additional dimensions spherically compactified. The manifold topology is $R \times S^3 \times R^3$, which is a product of parallelizable manifolds. Following the scheme (40) we will have now

$$e^a_\mu = \left(\begin{array}{c|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & & & \\ 0 & & \mathbb{M}_3 & & & \mathbb{O} & \\ 0 & & & & & & \\ \hline 0 & & & & a_0 & 0 & 0 \\ 0 & & \mathbb{O} & & 0 & a_0 & 0 \\ 0 & & & & 0 & 0 & a_0 \end{array}\right), \tag{44}$$

where now $\mathbb{M}_3$ is the matrix representing the paralellization of S^3. A global basis for $\mathcal{T}(S^3)$ have been obtained in Ref. [33], so we just summarize the result here. For $\mathbb{M}_3$ we have

$$\mathbb{M}_3 = a_1(t)\begin{pmatrix} c(\theta) & -s(\psi)c(\psi)s(\theta) & -s^2(\psi)s^2(\theta) \\ s(\theta)c(\phi) & s(\psi)[s(\psi)s(\phi) + c(\psi)c(\theta)c(\phi)] & s(\psi)s(\theta)[s(\psi)c(\theta)c(\phi) - c(\psi)s(\phi)] \\ s(\theta)s(\phi) & s(\psi)[c(\psi)c(\theta)s(\phi) - s(\psi)c(\phi)] & s(\psi)s(\theta)[c(\psi)c(\phi) + s(\psi)c(\theta)s(\phi)] \end{pmatrix}, \tag{45}$$

where we have abused by writing $s \equiv sin$ and $c \equiv cos$. We can just add here that $\mathbb{M}_3$ can be obtained from the *naive* triad $a_1(t)diag(d\psi, \sin\psi\, d\theta, \sin\psi\, \sin\theta\, d\phi)$ by means of a local Euler rotation (see [33] for details). It is a simple exercise to check that the frame (44) with $\mathbb{M}_3$ given by (45) give rise to the metric (43).

Having found the proper frames, we can proceed now to compute the motion equations. The vielbein (44) leads to the invariant

$$T = -6(H_0^2 + H_1^2 + 3H_0H_1 - a_1^{-2}). \tag{46}$$

The initial value equation then results

$$f + 12f'(H_0^2 + H_1^2 + 3H_0H_1) = 16\pi G\rho. \tag{47}$$

On the other hand, by varying respect the spatial sector of the vielbein, we get two equations:

$$2f'\left(7H_0^2 + 4H_1^2 + 9H_0H_1 + 3\dot{H}_0 + 2\dot{H}_1 - \frac{T}{3}\right) + 2f''\dot{T}(3H_0 + 2H_1) + f = -16\pi G p_1, \tag{48}$$

$$2f'\left(6H_0^2 + 9H_1^2 + 15H_0H_1 + 2\dot{H}_0 + 3\dot{H}_1\right) + 2f''\dot{T}(2H_0 + 3H_1) + f = -16\pi G p_0. \tag{49}$$

All the remaining equations are trivial or equal to these last two. This proof that the frame (44) is actually a parallelization of (43), for it leads to a consistent set of motion equations.

It is interesting to compare this compactification with the toroidal ones (in $D = 7$) worked in the last section. A remarkable property of the system (47)-(49) is that it admits a de Sitter expansion for the bulk while the extra

dimensions remains compactified with a constant scale factor, and that this happens for a pure vacuum. Actually, if we set H_0 as a constant and $H_1 = \rho = p_1 = p_2 = 0$, the system becomes (note that $T = -6(H_0^2 - a_1^{-2})$ is constant in this case)

$$f + 12H_0^2 f' = 0 \tag{50}$$

$$2f'(7H_0^2 - \frac{T}{3}) + f = 0, \tag{51}$$

equation (49) being equal to (50). Inserting the expression $T = -6(H_0^2 - a_1^{-2})$ and the value of f' from (50) in (51) we immediately get

$$H_0^2 = \frac{2}{3} a_1^{-2}, \tag{52}$$

irrespective of the functional form of $f(T)$. Under the present circumstances, every $f(T)$ model will assure relation (52). Note, however, that at this point the constant value of H_0 (or a_1) remains undetermined. Nevertheless, given that the observational evidence suggest that the size of the extra dimensions, if non null, should be very small at the present time, Eq. (52) invites to focus in the ultraviolet regime. This enable us to interpret inflation as a vacuum driven expansion given by the small compact extra dimensions. For this purpose let us invoke, as an example, the high energy deformation of GR with Born-Infeld structure worked in Refs. [3] and [4]. We have thus

$$f(T) = -\lambda\,[\sqrt{1 - 2\lambda^{-1}\,T} - 1], \tag{53}$$

where λ is the Born-Infeld constant. In this manner we have that Einstein gravity (in its absolute paralellism form) is recovered when $T/\lambda << 1$. The role of Eq. (50) will be to link the undetermined parameter H_0 (or a_1) with λ. Actually, replacing the functional form (53) in (50) we get the quadratic expression

$$(3u + 1)^2 = 1 + u, \qquad u = -6\lambda^{-1} H_0^2, \tag{54}$$

which leads to the solutions $H_0 = 0$ and $H_0^2 = 5\lambda/54$. Of course, $H_0 = 0$ just says that Minkowski spacetime is a vacuum solution of the theory. In turn, $H_0^2 = 5\lambda/54$, which is very closed to $H_0^2 = \lambda/12$, the value obtained in [4] for the inflationary era, represents a de Sitter accelerated expansion whose cause is the presence of the extra dimensions.

The remaining topology for the three additional dimensions is just $S^1 \times S^2$, so the full metric looks in this case

$$ds^2 = dt^2 - a_2^2(t)d\Theta^2 - a_1^2(t)(d\theta^2 + \sin^2\theta\; d\phi^2) - a_0^2(t)(dx^2 + dy^2 + dz^2). \tag{55}$$

We proceed now to find the parallel fields for the geometry (55). For this task we focus on the embedding of the submanifold $\mathcal{M}_3 = S^1 \times S^2$ in the four dimensional space $S^1 \times R^3$. Installing coordinates (Θ, X, Y, Z) in this manifold, we have that a parallelization of $\mathcal{T}^*(\mathcal{M}_3)$ reads

$$\begin{aligned} E^1 &= a_2(t) Z d\Theta - a_1(t)(Y dX - X dY) \\ E^2 &= a_2(t) Y d\Theta + a_1(t)(Z dX - X dZ) \\ E^3 &= a_2(t) X d\Theta - a_1(t)(Z dY - Y dZ). \end{aligned} \tag{56}$$

This base turns out to be a global parallelization for $S^1 \times S^2$ only if $a_1(t)$ and $a_2(t)$ are non null, so a singularity in the Riemannian sense (i.e. when at least one of $a_1(t)$ or $a_2(t)$ vanishes), represents actually a singularity in the parallelization process. In terms of the basis (Θ, θ, ϕ), where $X = \sin\theta\cos\phi$, $Y = \sin\theta\sin\phi$ and $Z = \cos\theta$ we have thus

$$\begin{aligned} E^1 &= a_2(t)\cos\theta\, d\Theta + a_1(t)\sin^2\theta\, d\phi \\ E^2 &= a_2(t)\sin\theta\sin\phi\, d\Theta + a_1(t)(\cos\phi\, d\theta - \sin\theta\sin\phi\cos\theta\, d\phi) \\ E^3 &= a_2(t)\sin\theta\cos\phi\, d\Theta - a_1(t)(\sin\phi\, d\theta + \cos\theta\sin\theta\cos\phi\, d\phi). \end{aligned} \tag{57}$$

In analogy with the analysis performed before, we can obtain now the motion equations. First it is necessary to compute the scalar invariant, which reads

$$T = -2(3H_0^2 + H_1^2 + 6H_0H_1 + 2H_1H_2 + 3H_0H_2 - a_1^{-2}). \tag{58}$$

Note that just the scale factor corresponding to S^2 is present in T, as must be the case. Moreover, the scalars (46) and (58) are different even when $a_0 = a_1$, because they represent different geometries. The full set of motion equations result

$$f + 4f'\left(H_1^2 + 3H_2H_0 + 3H_0^2 + 2H_1H_2 + 6H_1H_0\right) = 16\pi G\rho, \tag{59}$$

$$\begin{aligned} &f + 2f'\left(4H_1^2 + 2H_1H_2 + 9H_0^2 + 12H_1H_0 + 3H_0H_2 + 2\dot{H}_1 + 3\dot{H}_0\right) + \\ &+2f''\left(2H_1 + 3H_0\right)\dot{T} = -16\pi G\, p_2, \end{aligned} \tag{60}$$

$$\begin{aligned} &f + 2f'\left(-a_1^{-2} + 2H_1^2 + H_2^2 + 9H_0^2 + 3H_1H_2 + 9H_1H_0 + 6H_2H_0 + \dot{H}_1 + \dot{H}_2 + 3\dot{H}_0\right) + \\ &+2f''\left(H_1 + H_2 + 3H_0\right)\dot{T} = -16\pi G\, p_1, \end{aligned} \tag{61}$$

$$\begin{aligned} &f + 2f'\left(4H_1^2 + 6H_0^2 + H_2^2 + 4H_1H_2 + 10H_1H_0 + 5H_2H_0 + 2\dot{H}_1 + \dot{H}_2 + 2\dot{H}_0\right) + \\ &+2f''\left(2H_1 + H_2 + 2H_0\right)\dot{T} = -16\pi G\, p_0. \end{aligned} \tag{62}$$

A simple exercise shows that vacuum solutions with $H_1 = H_2 = 0$ and a non null constant H_0 doesn't exist at all for this specific compactification. Imposing these conditions on (59) and (60) we are lead to the inconsistent (when $H_0 \neq 0$) equations

$$\begin{aligned} f + 12f'H_0^2 = 0, \\ f + 18f'H_0^2 = 0. \end{aligned} \tag{63}$$

So, as far as vacuum driven inflation given by the small compact extra dimensions concerns, we see that the S^3 compactification is clearly favored.

B. Complete characterization of $D = 6$

In this subsection we consider the remaining topology (S^2) for the internal dimensions of the six dimensional manifold. We are ready now to go back to the metric (39). The topology of this manifold is now described by $R \times S^2 \times R^3$, so we proceed to parallelize this submanifold using a similar method to that employed in the end of the previous subsection. The key point is that we can unwrap the periodic coordinate Θ in (57) an to think about it as one of the bulk coordinates, lets say, x. In this way, we obtain the submanifold $\mathcal{M}_3 = S^2 \times R$. Again, embedding this so obtained three dimensional manifold in the four dimensional space $R^3 \times R$ and installing coordinates (X, Y, Z, x) on it, we obtain that a parallelization of $\mathcal{T}^*(\mathcal{M}_3)$ reads

$$\begin{aligned} E^1 &= a_0(t)\cos\theta\, dx + a_1(t)\sin^2\theta\, d\phi \\ E^2 &= a_0(t)\sin\theta\sin\phi\, dx + a_1(t)(\cos\phi\, d\theta - \sin\theta\sin\phi\cos\theta\, d\phi) \\ E^3 &= a_0(t)\sin\theta\cos\phi\, dx - a_1(t)(\sin\phi\, d\theta + \cos\theta\sin\theta\cos\phi\, d\phi). \end{aligned} \tag{64}$$

where again, $X = \sin\theta\cos\phi$, $Y = \sin\theta\sin\phi$ and $Z = \cos\theta$ was used. An observation is in order; nothing prevent to think on the unwrapped Θ periodic coordinate now as the y or z coordinate of the bulk, due to the isotropy and homogeneity of the latter. This actually means that we obtained three different but equivalent sets of parallel vector fields for (39).

With this vielbein in mind we can obtain the following torsion invariant and fields equations,

$$T = -2\left(-a_1^{-2} + H_1^2 + 6H_0H_1 + 3H_0^2\right) \tag{65}$$

$$f + 4f'\left(H_1^2 + 6H_0H_1 + 3H_0^2\right) = 16\pi G\rho \tag{66}$$

$$f + 2f'\left(6H_0^2 + 10H_0H_1 + 4H_1^2 + 2\dot{H}_0 + 2\dot{H}_1\right) + 2f''\dot{T}\left(2H_0 + 2H_1\right) = -16\pi G p_0 \tag{67}$$

$$f + 2f'\left(-a_1^{-2} + 9H_0^2 + 9H_0H_1 + 2H_1^2 + 3\dot{H}_0 + \dot{H}_1\right) + 2f''\dot{T}\left(3H_0 + H_1\right) = -16\pi G p_1 \tag{68}$$

Again, we are interested in solutions with $\rho = p_0 = p_1 = H_1 = 0$ and constant H_0. From (66) we get immediately

$$f + 12f'H_0^2 = 0, \tag{69}$$

which is consistent with (67). This expression replaced in (68) conduces to

$$H_0^2 = \frac{1}{3}a_1^{-2}.$$

This last equation is the analogue of (52). Note that this value of H_0 yields $T = 0$ by using (65), which turns out to be problematic. Actually, every ultraviolet deformation looks $f(T) = T + O(T^2)$, so we have $f(0) = 0$, $f'(0) = 1$, and then (69) is satisfied only when $H_0 = 0$. We can conclude in this way that physically admissible models do not contain $H_1 = 0$ and a non null constant H_0 as a solution.

V. CONCLUDING COMMENTS

This work was devoted to the study of spacetimes with topology $\mathrm{FRW}^4 \times \mathcal{M}^{D-4}$ in the context of the new gravitational schemes with absolute parallelism know as $f(T)$ theories. We have assumed that FRW^4 represent the spatially flat Friedmann-Robertson-Walker manifold corresponding to the bulk four dimensional spacetime of the standard cosmology, while $\mathcal{M}^{D-4}$ refers to the (D-4)-dimensional compact extra dimensions constituting the internal space. In one hand we focus the analysis on (D-4)-dimensional manifolds consisting of (D-4) copies of the torus, and on the other hand, several spherical compactifications were considered. While the obtention of the proper parallel vector fields for the toroidal compactifications realized in section III is trivial, for the (D-4) torus $T^{D-4} = S^1 \times ... \times S^1$ is a product of trivial parallelizable manifolds, the corresponding characterization of the spherical topologies results highly non trivial. Several aspects of this last issue were discussed in section IV, where the problem was possed in detail. Then, we proceed to characterize the six and seven dimensional cases, by finding the proper vielbeins for $\mathcal{M}^2 = S^2$ and $\mathcal{M}^3 = S^1 \times S^2$, $\mathcal{M}^3 = S^3$ respectively. This vielbeins constitute the starting point for *any* $f(T)$ cosmological model of compact extra dimensions, because they represent the basis responsible for the parallelization of the manifold under consideration, i.e., they define the spacetime structure. In both sections, we discuss a number of simple exact solutions of the field equations. One of the simple solutions explored throughout the different compactifications considered in this work concerns the existence of a constant scale factor for the compact dimensions while the bulk scale factor expands exponentially in time. This situation corresponds to a vacuum-driven inflation powered by the presence of the extra dimensions. We have shown that for five and six spacetime dimensions, this kind of behavior does not exist at all. In turn, this problem finds a solution in $D = 7$ but not for any kind of compactification, just for S^3. In this last case we found that for any $f(T)$ model, the relation between H_0 and a_1 (i.e., between the Hubble rate of the bulk and the constant scale factor of the extra dimensions), is given by equation $H_0^2 = \frac{2}{3}a_1^{-2}$. Therefore, Hubble rates characterizing the inflationary era find their hugeness in the exceedingly tiny value of a_1. We do not expect this exact solution to describe the whole details in the compactification process, but merely the asymptotic stage of it. We think that a solution for the internal dimensions of the form $a_{in} = a_1 + \alpha\, Exp(-H_0 t)$, with a constant α certainly do exist in the theory, though it probably will require the presence of matter fields and numerical techniques. However, the existence of the exact solution in $D = 7$ with S^3 topology for the extra dimensions lead us to think that a similar exact solution will exist *only* in $D = 11$, where the seven additional dimensions will be compactified with topology S^7. Note that S^7 is the only remaining parallelizable sphere, so vielbein fields with product structure $\mathrm{FRW}^4 \times S^7$ will constitute a parallelization of the entire manifold. Regarding this matter, we have proven in section IV that toroidal compactifications certainly do not lead to this kind of solution in any spacetime dimension D. It will be matter of future works to find the parallel vector fields of $\mathrm{FRW}^4 \times S^7$ and those of the other admissible eleven dimensional spherical topologies, and confirm (or dismiss) this conjecture. The veracity of this statement would permit to tend interesting linkages between this $f(T)$ deformed gravitational schemes and the low energy limit of M-theory.

Much more work is certainly necessary in order to fully understand the peculiarities present in the parallelization process lying behind $f(T)$ theories. In particular, the characterization of the whole spherical topologies in a cosmological context for a given spacetime dimension (with $D > 7$), remains entirely open. We like to think that our work constitute a first and significative step towards that goal.

Acknowledgments

F.F. is indebted with F. Canfora for many enlightening discussions about extra dimensions in cosmology. F.F. is supported by CONICET and Universidad de Río Negro. Y. V. is supported by FONDECYT grant 11121148, and by Dirección de Investigación y Desarrollo, Universidad de la Frontera, DIUFRO DI11-0071. P.G. and Y.V. acknowledge the hospitality of the Centro Atómico Bariloche where part of this work was made.

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